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GEOMETRIC PROPERTIES OF HOMOGENEOUS VECTOR FIELDS OF DEGREE TWO IN R^3

BY

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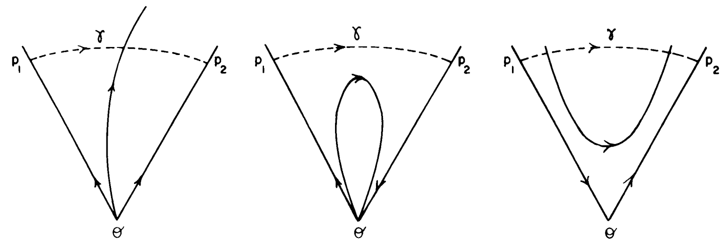
ABSTRACT. In the space of homogeneous polynomial vector fields of degree two, those that project on Morse-Smale vector fields on S^2 by the Poincaré central projection form a generic subset. The classification of those vector fields on S^2 without periodic orbits is given and applications to the study of local actions of the affine group of the line are derived.

Homogeneous polynomial vector fields of degree two in two variables $Q(x) = (Q_1(x), Q_2(x))$, $x \in R^2$, were thoroughly studied by J. Argemi [2] and L. Markus [6]. They classify under topological equivalence those vector fields presenting an isolated singularity at $0 \in R^2$. Recently, G. Tavares [10] studied quadratic polynomial vector fields in R^2. He classifies those vector fields which are structurally stable and without limit cycles. Here, we consider quadratic homogeneous polynomial vector fields in three variables, $Q(x) = (Q_1(x), Q_2(x), Q_3(x))$, $x \in R^3$. Any such vector field induces naturally on S^2 a tangent vector field Q_T defined as

$$Q_T(x) = Q(x) - (x_1Q_1(x) + x_2Q_2(x) + x_3Q_3(x))\cdot x, \qquad x = (x_1, x_2, x_3) \in S^2.$$

In other words, $Q_T(x)$ is the component of $Q(x)$ tangent to S^2 in x. Since the vector field Q is homogeneous, for any orbit $\gamma \subset S^2$ of Q_T, the surface $S(\gamma) = \{tp; t \in R, p \in \gamma\}$ is invariant by Q. We call these surfaces $S(\gamma)$ invariant cones of Q. By the Poincaré-Bendixson theory [3], the limit sets of the orbits of Q_T are singularities, closed orbits or graphs, provided that the singularities of Q_T are isolated. So the surfaces $S(\gamma)$ accumulate on rays, closed invariant cones or cones based on a graph on S^2. Homogeneous vector fields of degree $k \geqslant 2$ in R^3 were studied by Sh. R. Sharipov [9] and C. Coleman [4]. They determined the behavior of Q on the invariant cones. For example if γ is an orbit of Q_T having as α- and ω-limit sets two different singularities p_1 and p_2, respectively, then one has the following possibilities for Q:



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The purpose of this work is to prove the following theorems:

THEOREM 1. *Any quadratic homogeneous polynomial vector field Q in R^3 can be approximated by a vector field $\bar{Q}$ homogeneous of degree two inducing in S^2 a Morse-Smale vector field $\bar{Q}_T$* [7].

THEOREM 2. *There are seven different topological classes of Morse-Smale vector fields Q_T with no limit cycles.*

These results can be seen as a first step toward the classification of the quadratic homogeneous polynomial vector fields in R^3. For the polynomial vector fields that appear in this work we will use the topology of the coefficients, i.e. two polynomial vector fields are close if their coefficients are close.

These theorems have direct application in the study of some local actions of noncompact, nonabelian Lie groups as follows. Homogeneous vector fields Q imbed naturally in actions of the Lie group G^2 of affine transformations of the real line. These actions are defined by the vector fields as $X = I/(1 - k)$ and $Y = Q$, where $I(x) = x$ and Q is homogeneous of degree $k \geqslant 2$ in R^n, since $[I/(1 - k), Q] = Q$.

Let A be the set of G^2-actions defined by $(I/(1 - k), Q)$ in R^3 where $k = 2$, then

THEOREM 3. *There is a subset $\Sigma_A \subset A$ dense in A, whose actions are locally C^r-structurally stable, $r \geqslant 2$.*

In other words, Theorem 3 says that if $\phi \in \Sigma_A$ is defined by $(-I, Q)$ then every G^2-action ψ defined by vector fields (X, Y), $[X, Y] = Y$, such that X is C^r close to $-I$ and Y C^r close to Q, $r \geqslant 2$, is topologically equivalent to ϕ.

THEOREM 4. *There are seven different topological classes of G^2-actions belonging to Σ_A whose orbits are simply connected.*

We remark that for any action in Σ_A its linear part does not determine its topological type. In fact the orbits of the linear part all have dimensions less than 2, while the orbits of dimension 2 of the nonlinear action fill a nonempty open subset.

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1. Homogeneous polynomial vector fields of degree two in R^3. The aim of this section is to prove Theorem 1. The main difficulty in this proof is the stabilization of closed orbits of Q_T, i.e., the approximation of Q by $\bar{Q}$ such that all closed orbits of $\bar{Q}_T$ are hyperbolic. To stabilize a closed orbit of Q_T we first show that it is convex in the induced metric of S^2. Once this is proved it will be enough to find among the admissible perturbations a vector field transverse to the periodic orbit.

1.1. *Closed orbits of Q_T.* We prove here that if C is a closed orbit of Q_T there is a neighborhood $V_\epsilon(C)$ of C in S^2 and a vector field Q_μ close to Q such that Q_{μ_T} has in $V_\varepsilon(C)$ a hyperbolic closed orbit. The vector field Q in the usual basis in R^3 is:

$$Q(x) = (Q_1, Q_2, Q_3)(x), \qquad x = (x_1, x_2, x_3) \in R^3,$$

$$Q_1(x) = \sum_{\substack{i \leq j \\ i,j=1,2,3}} a_{ij} x_i x_j, \qquad Q_2(x) = \sum_{\substack{i \leq j \\ i,j=1,2,3}} b_{ij} x_i x_j, \qquad Q_3(x) = \sum_{\substack{i \leq j \\ i,j=1,2,3}} c_{ij} x_i x_j.$$

Define, $\Pi_0 = \{x \in R^3 | x_3 = 0\}$, $\Pi_i = \{x \in R^3 | x_i = 1\}$, $i = 1, 2, 3$, and the vector field W_Q in Π_3 given by:

$$W_Q(x) = (x_1 Q_3(x) - Q_1(x), \quad x_2 Q_3(x) - Q_2(x), 0) \quad \text{where } x = (x_1, x_2, 1).$$

By definition W_Q is tangent to the intersection of $S(\gamma) = \{tp | t \in R, p \in \gamma\}$ and Π_3, where γ is an orbit of Q_T. Moreover, $I(x)$ and $Q(x)$ are linearly dependent if and only if $W_Q(x) = 0$. Therefore the Poincaré central projection,

$$f_3 \colon \Pi_3 \to H_3^+ = \{(x_1, x_2, x_3) \in S^2 | x_3 > 0\},$$

where

$$f_3(x_1, x_2, 1) = (x_1^2 + x_2^2 + 1)^{-1/2}(x_1, x_2, 1)$$

is a diffeomorphism conjugating the flows induced by W_Q and $Q_T|_{H_3^+}$.

In a similar way we define vector fields R_Q in Π_1,

$$R_Q(x) = (0, x_2 Q_1(x) - Q_2(x), x_3 Q_1(x) - Q_3(x))$$

where $x = (1, x_2, x_3)$, and S_Q in Π_2,

$$S_Q(x) = (x_1 Q_2(x) - Q_1(x), 0, x_3 Q_2(x) - Q_3(x))$$

where $x = (x_1, 1, x_3)$. From now on all the propositions proved for W_Q will also be valid for R_Q and S_Q.

LEMMA 1. *Any straight line in Π_3 has at most two tangencies with the solutions of W_Q or it is a solution of W_Q.*

PROOF. For simplicity write, $W_Q(x) = (W_Q^1(x), W_Q^2(x))$ and let $p = (x_1, x_2, 1)$ be a contact point of W_Q with the line $ax_1 + b = x_2$. Then

$$W_Q^2(p) = aW_Q^1(p) \Leftrightarrow (x_2 Q_3 - Q_2)(p) = a(x_1 Q_3 - Q_1)(p)$$
$$\Leftrightarrow (x_2 - ax_1) Q_3(p) = Q_2(p) - aQ_1(p).$$

This last equation is of 2nd degree in x_1, so it has at most two real roots and this proves the lemma.

LEMMA 2. *Any great circle in S^2 has zero, two or four contacts with solutions of Q_T or it contains a solution of Q_T.*

PROOF. First of all $Q_T(x) = Q_T(-x)$ because $Q(x) = Q(-x)$. From this, obtain that if C is an orbit of Q_T then $-C$ is also an orbit of Q_T. So there is no closed orbit C such that if $x \in C$ then $-x \in C$. Then the lemma follows immediately from Lemma 1.

DEFINITION 1. We say that an orbit C of Q_T is convex if for any $x \in C$ the great circle tangent to C at x intersects C only at x, or C is entirely contained in the great circle.

We remark here that, by Lemma 1, the Poincaré central projection sends convex orbits of Q_T to convex orbits of W_Q, R_Q, S_Q, and by Definition 1 a convex closed orbit is entirely contained in a hemisphere.

PROPOSITION 1. *Any closed orbit of Q_T is convex.*

PROOF. Let C be a closed orbit of Q_T. We prove first that C is contained in a hemisphere, then the proposition will follow from Lemma 1. Let p be a point in C such that in a neighborhood of p, C is on one of the hemispheres determined by the great circle α, tangent to C in p; let A be this hemisphere. We will proceed by contradiction supposing that p is not the only intersection of C and α. These intersections cannot be symmetric, because at symmetric points q, $-q$, $Q_T(q) = Q_T(-q)$. We first notice that $C \not\subset \bar{A}$; otherwise there would exist another tangency point $q \neq -p$, such that the arc $\widehat{pq}$ of α not containing $-p$, will contain a third contact point r of α with Q_T. In this case α will have at least 6 contact points p, q, r, and their symmetrics; this is absurd. So there is a first point q_1 where C gets out of A, and a last point q_2 where C enters A. We can assume that q_1 and q_2 are not on the same side of the diameter $\overline{p(-p)}$, because otherwise there would exist at least 8 contact points of C with α. By the same reason we assume that in the arc $\widehat{pq_1}$ (Figure 1) there is only one contact point r_1 and similarly in the arc $\widehat{pq_2}$ there is only one contact point r_2. If p, $-p$, r_1 and r_2 are the unique contact points between C and α, this means that $r_2 = -r_1$. After q_1, C enters A again at the point s. There are two possibilities: either s is in the arcs $\widehat{(-r_1)(-p)}$ or $\widehat{(r_1 p)}$ (Figure 2). In the first case we define a closed broken curve γ as the part of the trajectory C between q_2 and s and the arc $\widehat{sq_2}$; γ divides S^2 in two connected components, and there are points of $-C$ in both of them, for example $-q_2$ and $-p$, this yields a contradiction (Figure 2a). In the second case we define a closed broken curve γ' as the part of the trajectory C between p and s, and the arc $\widehat{ps}$; γ' divides S^2 in two connected components (Figure 2b); at the point s, the curve C enters in one of these components and this yields a contradiction since it cannot leave it again.

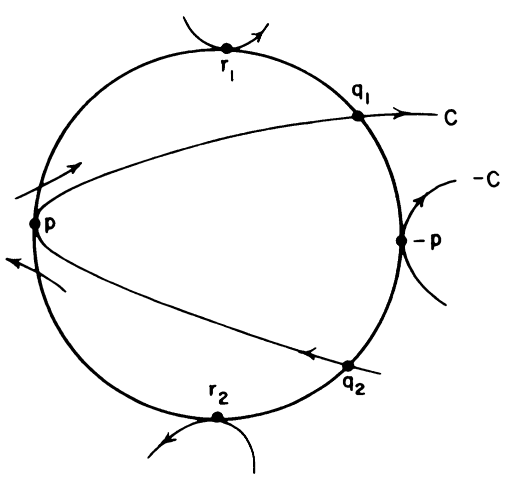

FIGURE 1

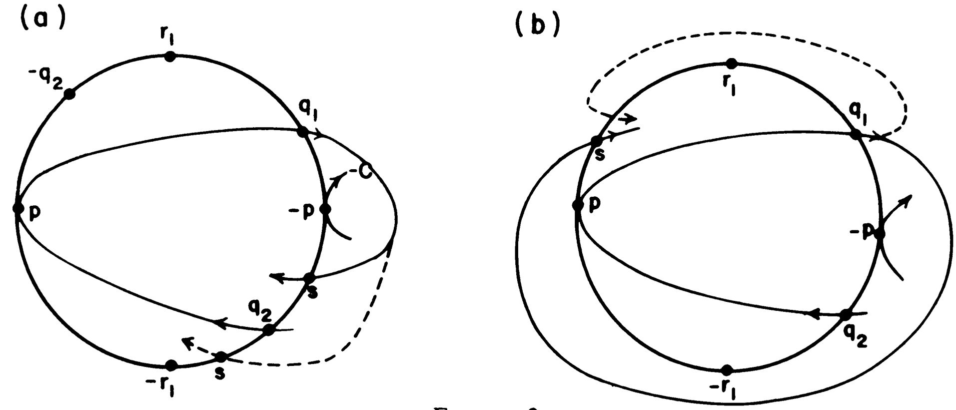

FIGURE 2

Since the closed orbits of Q_T are contained in hemispheres we can assume without loss of generality that they project through the central projection onto closed orbits of W_Q, R_Q, S_Q.

LEMMA 3. *Let γ be a closed orbit of W_Q. Then there is a polynomial vector field P in R^3 homogeneous of degree two such that W_P is transverse to γ.*

PROOF. Let (c, d) be a point in the disc bounded by γ. Define $P(x) = (cx_3^2 - x_1x_3, dx_3^2 - x_2x_3, 0)$. Then $W_P(x_1, x_2, 1) = (x_1 - c, x_2 - d, 0)$ and is clearly transversal to γ due to the convexity of γ.

From now on in this section, γ will be a closed orbit of W_Q and (c, d) a point in the interior of the disc bounded by γ. Take $Q_\mu = Q + \mu P$, where $0 < |\mu| < \mu_0$, μ_0 to be determined, and P as in Lemma 3. Then $W_{Q_\mu}(x_1, x_2, 1) = W_Q(x_1, x_2, 1) + \mu(x_1 - c, x_2 - d, 0)$. Let L be a local cross section to W_Q passing through a point $x_0 \in \gamma$. We call $\mathcal{P}$ the Poincaré map of W_Q, defined in an open set $L' \subset L$, with values in L. For μ_0 small enough all mappings $\mathcal{P}_\mu$ will be defined in L'. The expression of the Poincaré map $\mathcal{P}_\mu(x)$ can be found in [1, p. 381]: $\mathcal{P}_\mu(x) = u_{10}x + u_{01}\mu + u_{20}x^2 + u_{02}\mu^2 + \dots$. The expression of u_{01} is as follows. Let $x_1(t) = \phi(t)$, $x_2 = \psi(t)$, the solution of W_Q corresponding to γ. Then

$$u_{01} = \frac{1}{c}\exp\left(\int_0^\tau\left(\frac{\partial W_Q^1}{\partial x_1} + \frac{\partial W_Q^2}{\partial x_2}\right)ds\right)\cdot\int_0^\tau\left(\exp\left(-\int_0^s\left(\frac{\partial W_Q^1}{\partial x_1} + \frac{\partial W_Q^2}{\partial x_2}\right)dt\right)\right)A(s)\,ds,$$

where τ = period of γ, $c = -((d\phi/ds)^2 + (d\psi/ds)^2)|_{s=0}$, and $A(s) = (\psi(s) - d)W_Q^1(s) - (\phi(s) - c)W_Q^2(s)$.

Let $n \geqslant 1$ be the least integer such that $d^{(n)}(\mathcal{P}(x) - x|_{x=x_0}) \neq 0$ and $d^{(n-1)}(\mathcal{P}(x) - x|_{x=x_0}) = 0$. If n is even, the curve γ is attracting on one side and expanding on the other. In this case we say that γ has even multiplicity. If n is odd, γ is attracting or expanding on both sides and we say that γ has odd multiplicity. The following Lemmas 4 and 5 can be found in [1, pp. 397–398].

LEMMA 4. *Let n be an even number and $u_{01} \neq 0$. Then if $(u_{01})/(u_{n0}) > 0$, the equation $\mathcal{P}_\mu(x) = x$ has two different real roots for $\mu > 0$ ($\mu < 0$), which are simple roots and has no real roots for $\mu < 0$ ($\mu > 0$).*

LEMMA 5. *If n is odd and $u_{01} \neq 0$, the equation $\mathcal{P}_\mu(x) = x$ has precisely one real root for both $\mu > 0$ and $\mu < 0$, and this root is simple.*

LEMMA 6. *The coefficient u_{01} of the Poincaré map $\mathcal{P}_\mu(x)$ of the vector field W_{Q_μ} is different from zero.*

PROOF. It is enough to show that $A(s) \neq 0$. If $\langle\ ,\ \rangle$ denotes the inner product of R^2 and $X^\perp$ the vector field orthogonal to X, we have

$$A(s) = \langle((\phi(s) - c), (\psi(s) - d))^\perp, W_Q(s)\rangle$$
$$= \langle W_P^\perp(x(s)), W_Q(x(s))\rangle, \qquad x(s) \in \gamma.$$

But $|W_P(x)^\perp| \cdot |W_Q(x)|\cos\theta = 0$ if and only if $\cos\theta = 0$ and this implies that W_P and W_Q are linearly dependent. This is absurd due to Lemma 3.

Notice that if $x' \in L'$ is a simple root of the equation $\mathcal{P}_\mu(x) = x$, then the closed orbit of W_{Q_μ} is hyperbolic. Using the last three lemmas and the remark above, we immediately obtain the following propositions:

PROPOSITION 2. *Let γ be a closed orbit of even multiplicity of W_Q. There is $\varepsilon > 0$, $\mu_0 > 0$ with the following property: either for every $\mu > 0$, $|\mu| < \mu_0$, W_{Q_μ} has precisely two closed orbits in $V_\varepsilon(\gamma)$ which are structurally stable and has no closed orbits in $V_\varepsilon(\gamma)$ for every $\mu < 0$, $|\mu| < \mu_0$, or, conversely, for every $\mu > 0$, $|\mu| < \mu_0$, W_{Q_μ} has no closed orbits in $V_\varepsilon(\gamma)$ whereas for $\mu < 0$, $|\mu| < \mu_0$, W_{Q_μ} has precisely two closed orbits in $V_\varepsilon(\gamma)$ which are hyperbolic.*

PROPOSITION 3. *If γ is a closed orbit of odd multiplicity of W_Q, there exist $\varepsilon > 0$, $\mu_0 > 0$ such that for all $\mu \neq 0$, $|\mu| < \mu_0$, W_{Q_μ} has a single closed orbit in $V_\varepsilon(\gamma)$ which is hyperbolic.*

1.2. *The saddle connections of Q_T.* Here we will prove two lemmas that will be necessary for the proof of Theorem 1.

LEMMA 7. *The saddle self-connections of Q_T are convex.*

PROOF. Let s be a saddle point with a self-connection l; we take a great circle α through s, such that in s, l is locally in one side of α. The local behavior of Q_T at s is similar to the case of a tangency point of a closed orbit in Proposition 1, where it was proved that the closed orbit is entirely contained in one of the hemispheres determined by a great circle α' near α. The same arguments can be repeated here to show that l is entirely contained in one hemisphere. By Lemma 1, l is convex.

LEMMA 8. *Let Q_T be a vector field without saddle self-connections; if s_1 and s_2 are two different saddle points with a common separatrix l, then l is a convex orbit of Q_T.*

PROOF. Let l be the unstable manifold of s_1, and α the great circle through s_1 and s_2. We have two possibilities:

(a) $s_2 = -s_1$. By Lemma 2 we can assume that the number of intersections between α and l is at most 2. We prove first that l cannot intersect α transversely in a unique point. In fact in this case the symmetry of Q_T in neighborhoods of s_1 and $-s_1$ will imply that the unstable manifold of s_1 is locally on one side of α (see Figure 3a). This is absurd because near α there is α' with 8 contact points with $\mathcal{Q}_T$.

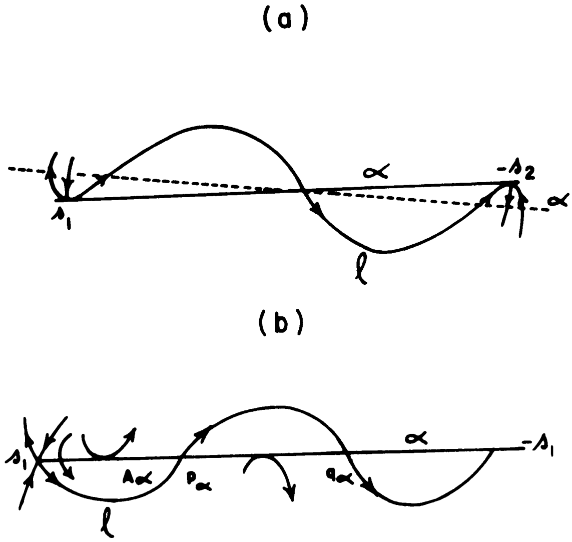

FIGURE 3

Now we suppose that α intersects l in 2 points, p_α and q_α, where p_α is the first one in the positive direction of l; let A_α be the region limited by l and α between s_1 and p_α; as $s_2 = -s_1$ there is an infinite number of great circles α through s_1 and $-s_1$, such that α intersects l in 2 points and Q_T enters in A_α through α near s_1; in this case α has at least 6 contact points with Q_T. We just proved that α cannot intersect l in 2 points. As there is always a great circle α_0 which intersects l, we conclude that l is entirely contained in α_0; then it is a convex orbit of Q_T.

(b) $s_2 \neq -s_1$. By Lemma 2 we can assume that the number of intersections between α and l is 0 or 1; we begin supposing that q is the unique intersection point of l with α; in this case by Lemma 2 l must be on both sides of α; we call A the region limited by l and α between s_1 and q, and B the region limited by l and α between q and s_2 (Figure 4). It is easy to prove that there is a great circle α' near α with at least 6 contact points in any one of the following situations:

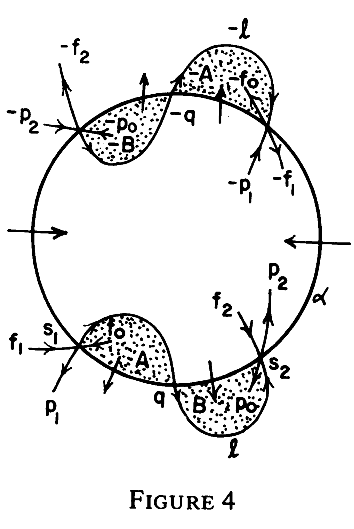

FIGURE 4

(a) α is tangent to the stable or unstable manifold of s_1 or s_2;

(b) the stable manifold of s_1 does not intersect A or it does intersect A but is not entirely contained on it.

(c) the unstable manifold of s_2 does not intersect B or it does intersect B but is not entirely contained on it;

(d) s_1, s_2, and their symmetric points are not the only contact points of Q_T with α.

We can suppose then that we have a vector field Q_T where none of the situations above appear. By Lemma 1, §2, Q_T has at most 4 sinks and 4 sources radially opposite. As f_0, $-p_0$ are on the same hemisphere, then $f_0 \neq -p_0$. On the other hand, since by hypothesis Q_T has no saddle self-connections, Q_T will have at most 4 sinks and 4 sources radially opposite if and only if $f_2 = -p_2$ and $f_1 = -p_1$. If $f_1 = -p_1$, we call γ the closure of the union of the unstable separatrix of s_1, between s_1 and p_1, and the unstable separatrix of $-s_1$, between $-s_1$ and p_1 (Figure 5); $\gamma \cup (-\gamma)$ divides S^2 in two symmetric connected components; this implies that $f_2 \neq -p_2$. So we can conclude that l cannot intersect α, and then s_1, s_2 and l are entirely contained in one hemisphere. The problem is then reduced to studying the contact points of saddle connections in the plane. We can suppose without loss of generality that s_1, s_2 and l are projected on Π_3 by the central projection f_3. Let $f_3(s_1)$, $f_3(s_2)$ and $f_3(l)$ be these projections. By hypothesis, the line $f_3(\alpha) = f_3(s_1) \cdot f_3(s_2)$, through $f_3(s_1)$ and $f_3(s_2)$, leaves $f_3(l)$ entirely contained in a half plane. By Lemma 1, for every $p \in f_3(l)$, the straight lines $p \cdot f_3(s_1)$, $p \cdot f_3(s_2)$ intersect $f_3(l)$ only at p. This implies that $f_3(l)$ is a convex curve.

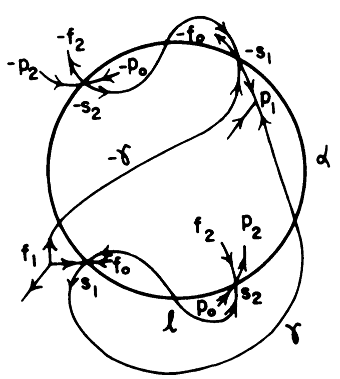

FIGURE 5

1.3. *The proof of Theorem* 1. We will prove this theorem in three steps:

(a) We approximate Q_T by Q_T' whose singularities are all hyperbolic.

Consider the vector field $W_Q = (W_Q^1, W_Q^2)$ as in 1.1. There exists a vector field Q^1, close enough to Q such that $W_{Q^1} = (W_{Q^1}^1, W_{Q^1}^2)$ has a finite number of singularities. This can be obtained by using Sard's theorem on $Q^1 = Q + P^1$, where $P^1(x_1, x_2, x_3) = (\varepsilon_1 x_3^2, \varepsilon_2 x_3^2, 0)$ for $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$, small. Now we consider $P^2(x_1, x_2, x_3) = (\delta x_1 x_3, \delta x_2 x_3, 0)$ and $Q^2 = Q^1 + P^2$. By definition,

$$W_{Q^2}(x_1, x_2, 1) = W_{Q^1}(x_1, x_2, 1) + W_{P^2}(x_1, x_2, 1) = W_{Q^1}(x_1, x_2, 1) - \delta(x_1, x_2, 1)$$

and $$DW_{Q^2}(x_1, x_2, 1) = DW_{Q^1}(x_1, x_2, 1) - \delta I(x_1, x_2, 1),$$

where I is the identity map. For $\delta > 0$ small enough, the set of singularities of W_{Q^2} will be close to the set of singularities of W_{Q^1}, and since the derivatives are close, i.e., $DW_{Q^2}(x) = DW_{Q^1}(x) - \delta I(x)$, for every x in Π_3, we choose a $\delta > 0$ small, such that the singularities of W_{Q^2} will also be hyperbolic. From the vector field Q^2, we repeat the same argument for the vector fields induced on Π_1 and Π_2. This provides us with a vector field Q'_T close to Q^2_T whose singularities are all hyperbolic.

(b) We approximate Q'_T by Q''_T without saddle connections.

Take Q' the vector field obtained in (a); as we know, Q'_T has only a finite number of saddle connections. Let l be a saddle self-connection of Q_T and s the saddle point; by Lemma 7 we can suppose that s and l are in H_3^+; $f_3(l)$ and $f_3(s)$ are the central projection of l and s in Π_3; as l is a convex orbit (Lemma 7), so is $f_3(l)$ because the Poincaré central projection preserves convexity. Therefore there is a point (c, d) such that the vector field $(x_1 - c, x_2 - d)$ is transverse to $f_3(l)$. Now if we take $P^3(x_1, x_2, x_3) = (x_3x_1 - cx_3^2, x_3x_2 - dx_3^2, 0)$ and $Q^3 = Q' \pm \delta P^3$, for $\delta > 0$ small enough, Q^3_T will not have saddle self-connection at s because $(x_1 - c, x_2 - d)$ is obtained by central projection from the vector field P^3. So we approximate Q' by a vector field Q^4 such that Q^4_T does not have saddle self-connections; now using Lemma 8 and analogous arguments we approximate Q^4 by a vector field Q'' such that Q''_T does not have saddle connections.

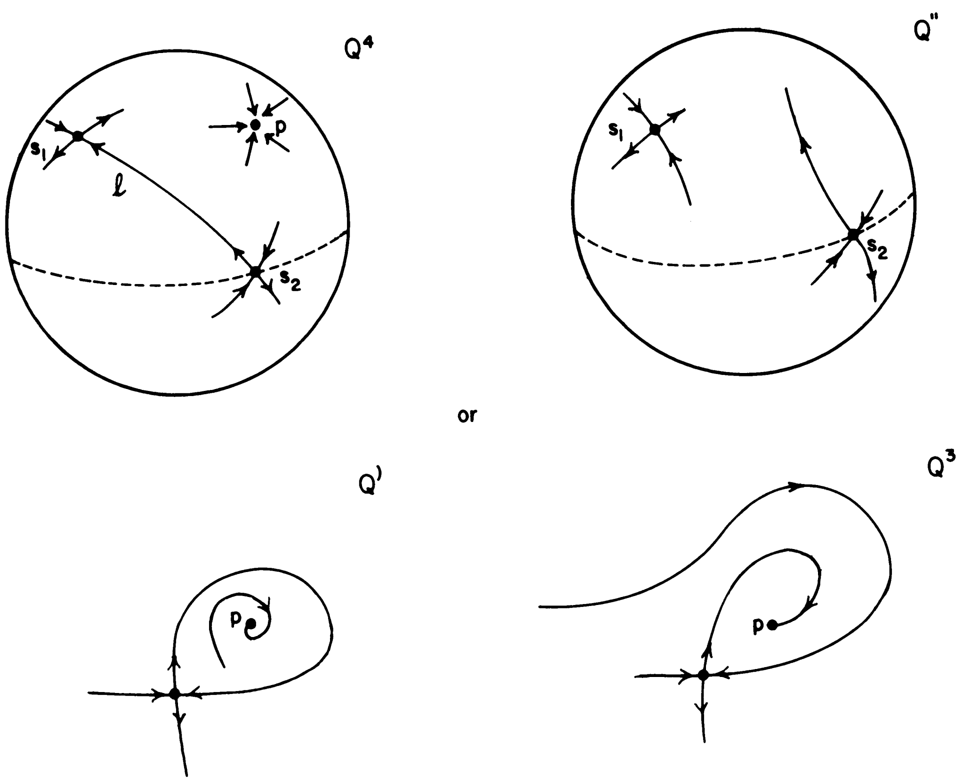

FIGURE 6

(c) We approximate Q''_T by $\overline{Q}_T$ whose closed orbits are hyperbolic.

Take Q''_T the vector field obtained in (b); this we know has only hyperbolic singularities, no saddle connections and, since Q''_T is analytic, a finite number of closed orbits. Using 1.1, there exists a homogeneous vector field $\overline{Q}_T$ close enough to Q''_T such that all closed orbits are hyperbolic. This finishes the proof of Theorem 1.

2. Classification of the vector fields Q_T, structurally stable and without limit cycles. The vector fields Q_T and Q'_T in S^2 induced by Q and Q' are said to be topologically equivalent if there exists a homeomorphism h in S^2 sending orbits of Q_T onto orbits of Q'_T. Here we will exhibit all possible topological equivalence classes of vector fields Q_T in S^2, where Q_T is Morse-Smale without closed orbits. A classification of the vector fields Q_T that exhibit closed orbits involves considerations related with Hilbert's 16th problem **[5]**, **[8]**, **[10]**. In 2.1, we prove the existence of a symmetric continuous curve C in S^2, Q_T-invariant that divides S^2 in two symmetric regions H_1 and H_2 such that $Q_{T|H_1}$ is symmetric radially to $Q_{T|H_2}$ (i.e. $Q_T(x) = Q_T(-x)$). Then the problem is reduced to studying the vector field $Q_{T|\overline{H}_1}$. In 2.2, we determine the possible phase portrait of $Q_{T|\overline{H}_1}$; in 2.3 we exhibit for each possible topological type, a vector field Q_T.

2.1. *Construction of a symmetric continuous curve C in S^2, Q_T-invariant.* Since $Q_T(x) = Q_T(-x)$, for every $x \in S^2$, then $Q_T(x) = Q_T(-x) = 0$ implies that $DQ_T(x) = -DQ_T(-x)$, i.e., a sink opposes to a source and a saddle opposes to a saddle.

LEMMA 1. *The vector field Q_T has* 2, 6, 10 *or* 14 *singularities, radially opposite.*

PROOF. By the remark above we have that Q_T has $2s$ saddles, p sinks and p sources.. sing the fact that $2p - 2s = 2$, we have that the number of singularities of Q_T is $4s + 2$. We show now that Q_T has at most 14 singularities. In fact, suppose that Q has at least 6 radial solutions. Since these solutions are radially opposite by pairs, we can take them as a base in R^3. In this base $Q = (Q_1, Q_2, Q_3)$ is written as:

$$Q_1(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3,$$
$$Q_2(x_1, x_2, x_3) = b_{22}x_2^2 + b_{12}x_1x_2 + b_{13}x_1x_3 + b_{23}x_2x_3,$$
$$Q_3(x_1, x_2, x_3) = c_{33}x_3^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + c_{23}x_2x_3.$$

From the symmetry of Q_T we have: the number of the singularities of $Q_T = 2$ (the number of the singularities of W_Q) + (the number of the singularities of $Q_{T|\Pi_0}$).

$W_Q(x_1, x_2, 1) = (W_1, W_2, 0)(x_1, x_2, 1)$, where

$$\begin{aligned} W_1(x_1, x_2, 1) &= x_1^2(c_{13} - a_{11}) + x_1(c_{33} - a_{13}) \\ &\quad + x_1^2x_2c_{12} + x_1x_2(c_{23} - a_{12}) - x_2a_{23}, \\ W_2(x_1, x_2, 1) &= x_2^2(c_{23} - b_{22}) + x_2(c_{33} - b_{23}) \\ &\quad + x_1x_2^2c_{12} + x_1x_2(c_{13} - b_{12}) - x_1b_{13}. \end{aligned}$$

For $W_1 = 0$, we have

$$x_2 = \left(x_1^2(a_{11} - c_{13}) + x_1(a_{13} - c_{33})\right) / \left(x_1^2c_{12} + x_1(c_{23} - a_{12}) - a_{23}\right), \quad (1)$$

and for $W_2 = 0$, we have

$$x_1 = \left(x_2^2(b_{22} - c_{23}) + x_2(b_{23} - c_{33})\right) / \left(x_2^2 c_{12} + x_2(c_{13} - b_{12}) - b_{13}\right). \quad (2)$$

There are two cases to consider:

(a) $c_{12} \neq 0$. Replacing (1) in (2), we have a fifth degree equation in x_1, therefore W_Q has at most 5 singularities. But $Q_{T|\Pi_0}$ has at least 4 singularities which are those determined by the x_1- and x_2-axes. Assuming there is one more radial solution in the plane Π_0, say $\{x_2 = \lambda x_1, \lambda \neq 0\}$, then $Q_3(x_1, \lambda x_1, 0) = c_{12}\lambda x_1^2 = 0$, so $c_{12} = 0$, which is absurd. Thus Q_T has 14 singularities.

(b) $c_{12} = 0$. Since $Q_3(x_1, x_2, 0) = 0$, then $Q(x_1, x_2, 0) = (Q_1, Q_2, 0)(x_1, x_2, 0)$ is a quadratic homogeneous vector field in R^2. These vector fields have at most 3 radial solutions [2], [6], i.e., Q_T has at most 6 singularities in the plane Π_0 while W_Q has at most 4 singularities because it is a polynomial vector field of degree two. Thus Q_T has at most 14 singularities. Using that number of singularities of $Q_T = 4s + 2$, for $s = 0, 1, \ldots$ we have that Q_T has 2, 6, 10 or 14 singularities.

PROPOSITION 1. *If Q_T has 6, 10 or 14 singularities without closed orbits, then there is a symmetric continuous curve C in S^2, Q_T-invariant and with 6 singularities (2 saddles, 2 sinks, and 2 sources).*

PROOF. We will prove this proposition only when Q_T has 10 singularities, i.e., three sinks $\{p_1, p_2, p_3\}$, three sources $\{f_1 = -p_1, f_2 = -p_2, f_3 = -p_3\}$, and four saddles $\{s_1, s_2 = -s_1, s_3, s_4 = -s_3\}$. The other cases are analogous. Let $W^s(p) = \{q \in S^2 | \omega\text{-lim}(q) = p\}$ be the stable manifold of p, and $W^u(p) = \{q \in S^2 | \alpha\text{-lim}(q) = p\}$ the unstable manifold of p. Given an arbitrary saddle s_0 of Q_T, by the Poincaré-Bendixson theorem at least one of the invariant manifolds $W^s(s_0)$, $W^u(s_0)$ has for limit set two different singularities. From this we obtain that if Q_T has 10 singularities there exist two possibilities for two arbitrary saddles s_k, s_l different and not symmetric, $k, l \in \{1, 2, 3, 4\}$.

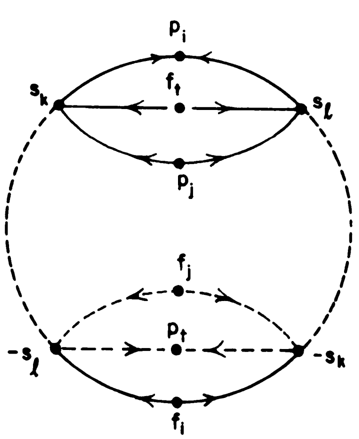

FIGURE 7

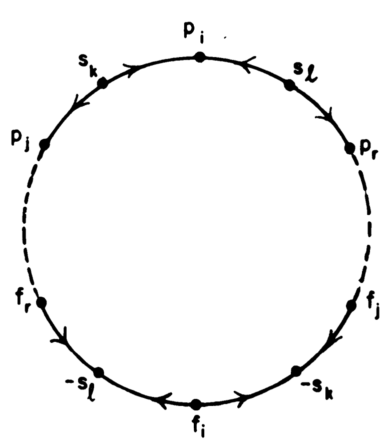

FIGURE 8

(a) ω-lim $W^u(s_k) = \omega$-lim $W^u(s_l) = \{p_i, p_j\}, i \neq j, i, j \in \{1, 2, 3\}$.

In this case it is easy to see that $\overline{W^s(s_k)} \subset \overline{W^s(p_i)}$ and $\overline{W^s(s_k)} \subset \overline{W^s(p_j)}$ (Figure 7). If $f_i \in \overline{W^s(s_k)}$ then $f_i \in \overline{W^s(p_j)}$. Let γ be an orbit of Q_T such that ω-lim $\gamma = p_j$ and α-limit $\gamma = f_i$. The curve C is then $\gamma \cup W^u(s_k) \cup \{-\gamma\} \cup W^s(-s_k)$. If $f_j \in \overline{W^s(s_k)}$ then $f_j \in \overline{W^s(p_i)}$. Now if γ is an orbit of Q_T such that ω-limit $\gamma = p_i$ and α-limit $\gamma = f_j$, then C is $\gamma \cup W^u(s_k) \cup \{-\gamma\} \cup W^s(-s_k) \ldots$.

(b) ω-limit $W^u(s_k) = \{p_i, p_j\}$, ω-limit $W^u(s_l) = \{p_i, p_r\}$, $i \neq j, r;\ j \neq r,\ i, j, r \in \{1, 2, 3\}$.

Here we have that $\overline{W^s(s_k)} \subset \overline{W^s(p_i)}$ and $\overline{W^s(s_k)} \subset \overline{W^s(p_j)}$ (Figure 8). If $f_i \in \overline{W^s(s_k)}$ then $f_i \in \overline{W^s(p_j)}$; if $f_j \in \overline{W^s(s_k)}$ then $f_j \in \overline{W^s(p_i)}$; if $f_r \in \overline{W^s(s_k)}$ then $f_r \in \overline{W^s(p_i)}$ and $f_r \in \overline{W^s(p_j)}$. In each case let γ be an orbit of Q_T such that γ satisfies, respectively: ω-limit $\gamma = p_j$ and α-limit $\gamma = f_i$; ω-limit $\gamma = p_i$ and α-limit $\gamma = f_j$; ω-lim $\gamma = p_i$ and α-limit $\gamma = f_r$. Then C is $\gamma \cup W^u(s_k) \cup \{-\gamma\} \cup W^s(-s_k)$ or $\gamma \cup W^u(s_l) \cup \{-\gamma\} W^s(-s_l)$. This finishes the proof.

2.2. *The phase portrait of the vector field Q_T.* Let H_1 and H_2 be the two connected symmetric components of $S^2 - C$. We will call α the restriction to $\overline{H}_1$ of a great circle of S^2. By Lemmas 1 and 2, we have the following possibilities: (a) if α contains a nontrivial solution of Q_T, then α is invariant by Q_T, i.e. it is a union of solutions of Q_T; (b) α has at most one contact with solutions of Q_T in H_1 if α passes through a singularity of C; (c) α has at most two contacts with solutions of Q_T in H_1. We will use the above to prove the following lemmas, recalling that in these proofs, α will always be transverse to the curve C.

LEMMA 2. *There is no Q_T with the following phase portrait for $Q_{T|\overline{H}_1}$.*

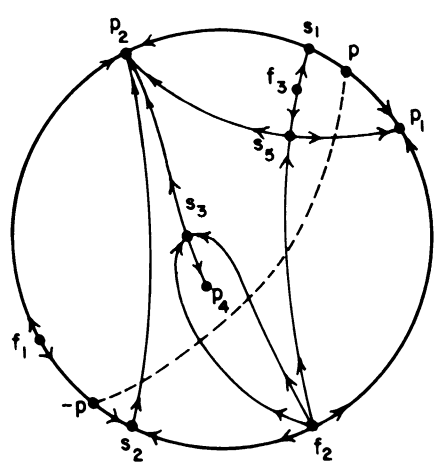

FIGURE 9

PROOF. We take in C close to s_1 a point p. Every great circle through p has at least two contacts in H_1 and those passing through s_3 or p_4 have three. Absurd

LEMMA 3. *There is no Q_T with the following phase portraits for $Q_{T|\overline{H}_1}$.*

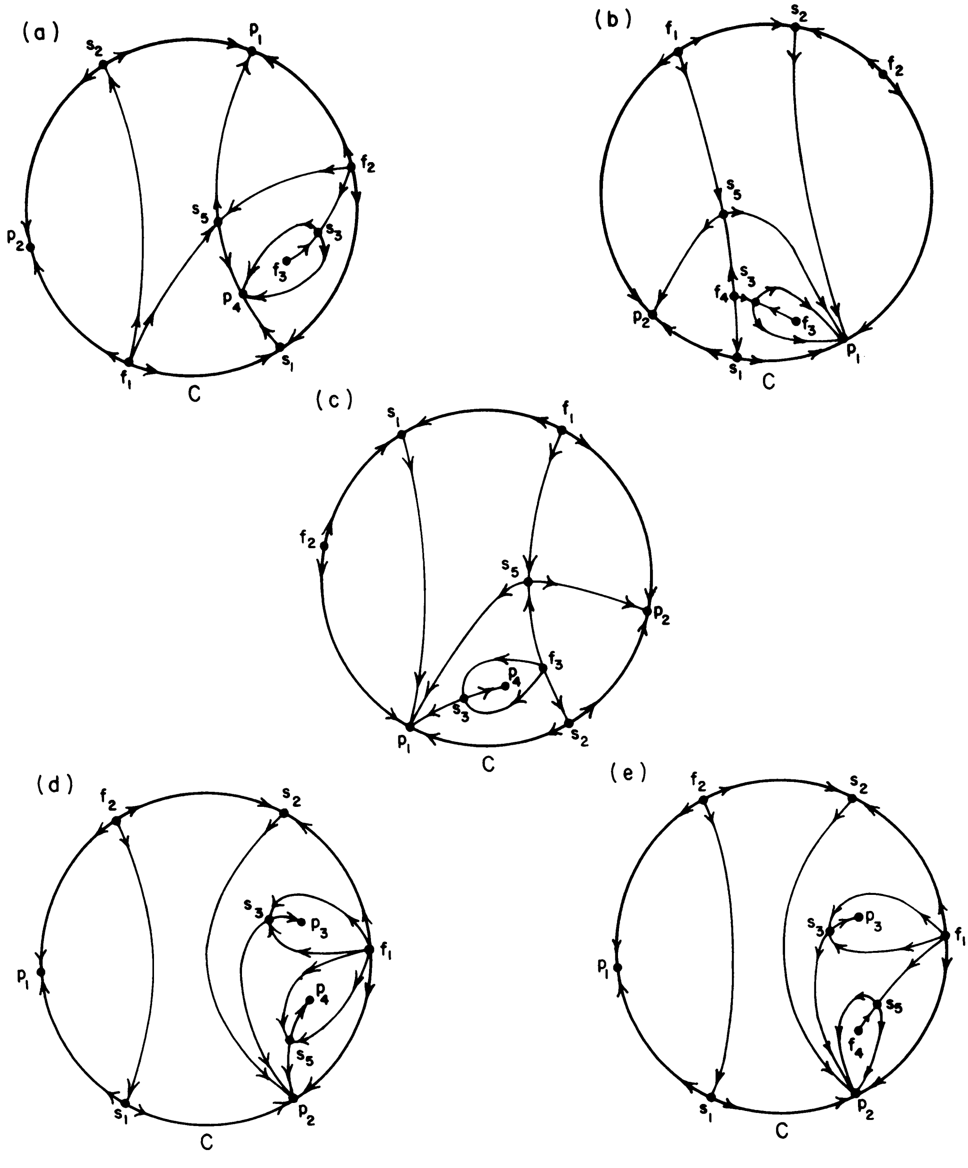

FIGURE 10

PROOF: *case* (a). We consider all the great circles through s_5; the proof is immediate if there is some singularity in the shadowed regions (Figures 11 and 12); suppose now that the singularities are in the regions A or A'; if some of them are in A it is enough to consider the great circle passing through this singularity and s_1 (so through s_2) (Figure 11). If all of them are in A', we consider the great circle passing through s_1, s_2 and f_3 or s_3 (Figure 12); this proves case (a). The other cases are analogous.

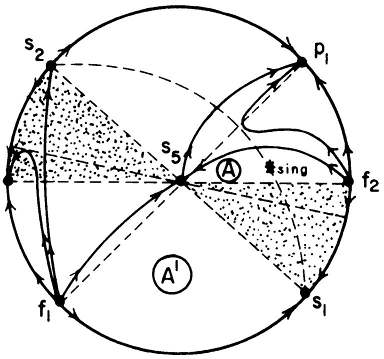

FIGURE 11

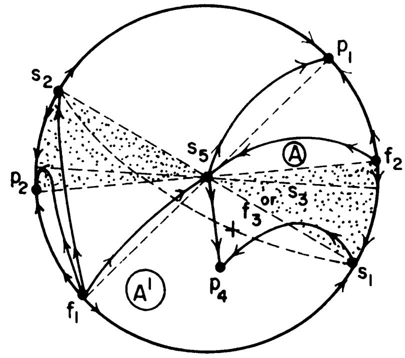

FIGURE 12

LEMMA 4. *There is no Q_T with the following phase portraits for $Q_{T|\bar{H}_1}$.*

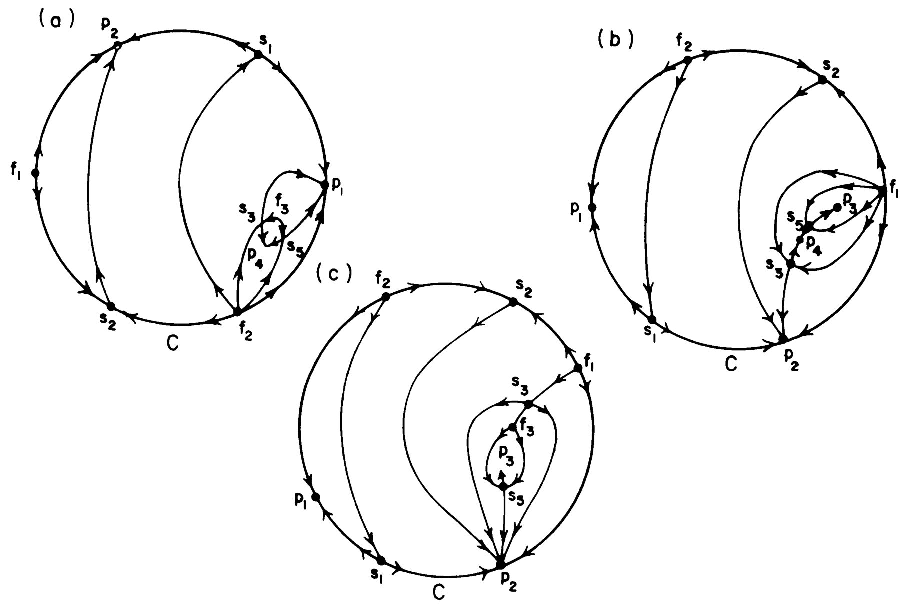

FIGURE 13

PROOF. Proceeding as in the last lemma, we are reduced to studying the case when the singularities are all in A' (Figure 14).

We can suppose without loss of generality that p_1 and f_1 are in the x_1-axis, p_2 and f_2 in the x_2-axis, s_3 in the x_3-axis and s_1 is not in the plane $x_3 = 0$. Let B: $R^3 \to R^3$ be a diagonal linear mapping such that $B(s_1)$ is close enough to a point s in the plane $x_3 = 0$, i.e., the coefficient c_{12} (see 2.1) of the homogeneous vector field $Q' = B \cdot Q \cdot B^{-1}$ is close to zero. Now, let W be a vector field in S^2, induced by a homogeneous vector field whose coefficients are equal to those of Q' except c_{12} that we take equal to zero. W leaves invariant the plane $x_3 = 0$ and is such that

$W|_{\bar{H}_1}$ is equivalent to a vector field of degree two in R^2 whose singularities at infinity correspond to the singularities of W in the plane $x_3 = 0$ [**10**]. If W is not Morse-Smale [**10**] there is W' close to W and so close to Q_T', such that W' is Morse-Smale; since Q_T' is Morse-Smale by hypothesis, we have that W' is equivalent to Q_T'. This is absurd [**10**].

DEFINITION 1. Two regions A and A' in the phase portrait of a vector field X are symmetrical if $X|_A$ and $X|_{A'}$ are topologically equivalent by an orientation reversing homeomorphism.

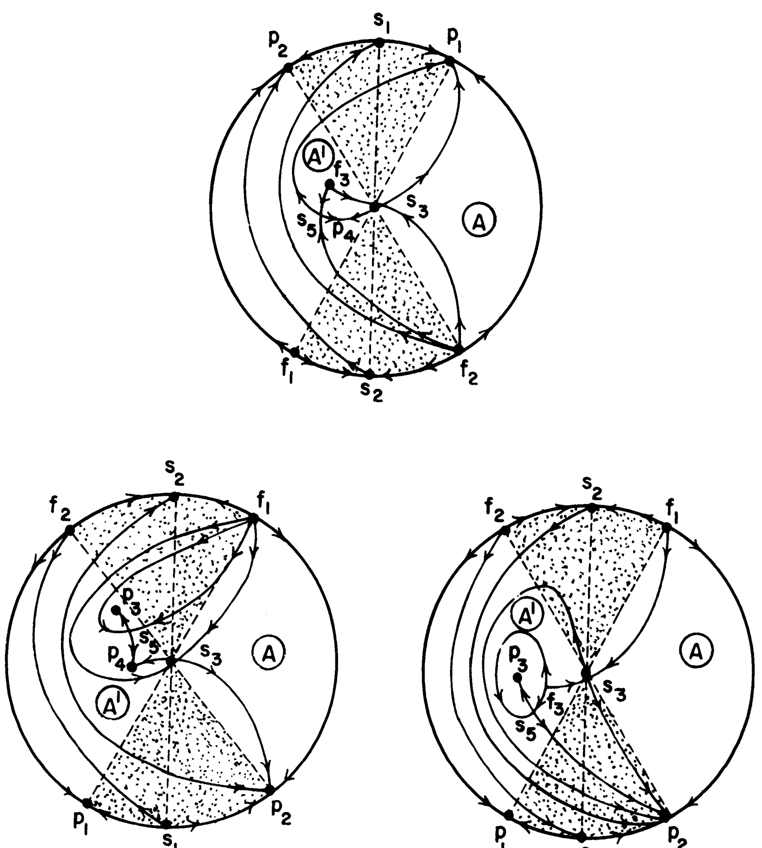

FIGURE 14

LEMMA 5. *For* $Q_{T|\bar{H}_1}$ *we have*: (a) *If the region* (A) (*Figure* 15) *and its symmetric occur in the phase portrait of* $Q_{T|\bar{H}_1}$, *then* Q_T *has no singularities in* H_1; (b) *If either region* (B) *or* (E) *occurs in the phase portrait of* $Q_{T|\bar{H}_1}$ *then the regions* (C), (D), *the symmetric of* (B) *and the symmetric of* (E), *cannot occur in the same phase portrait* (*Figure* 15).

PROOF. This proof can be found in [**10**, p. 630] or by using the same argument as in Lemma 3.

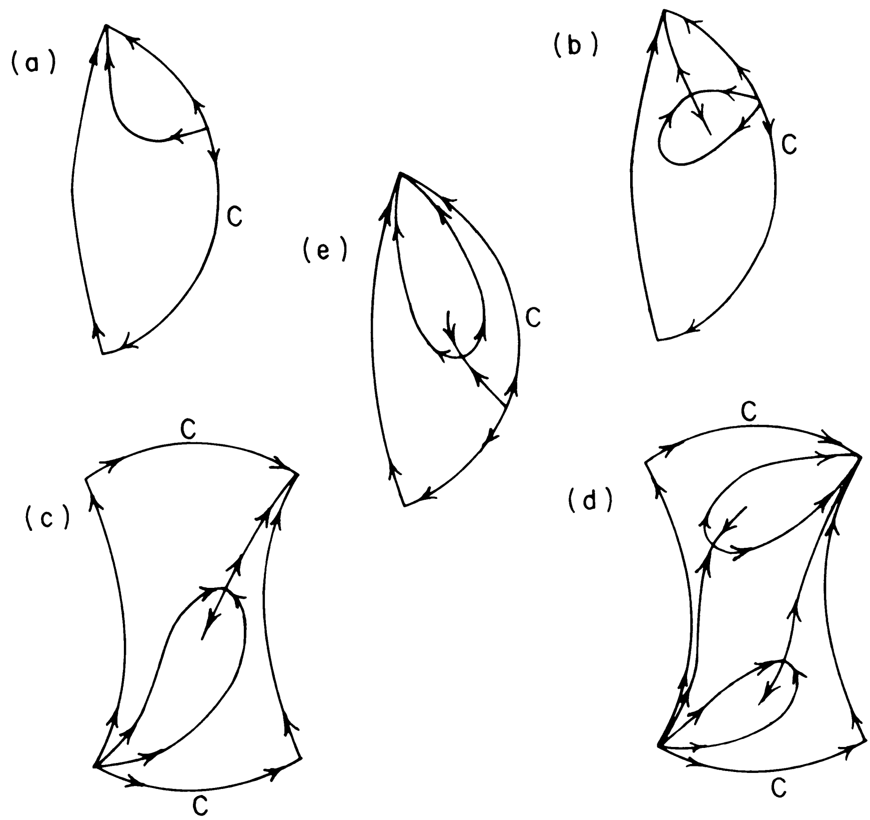

FIGURE 15

PROPOSITION 2. *If Q_T is a Morse-Smale vector field with* 6, 10, *or* 14 *singularities and no closed orbits, then $Q_{T|\bar{H}_1}$ has the following phase portraits*:

PROOF. It follows immediately from Lemmas 2–5.

2.3. **PROOF OF THEOREM** 2. In this context, if Q_T is equivalent to Q_T' ($Q_T \sim Q_T'$), we will say that Q is equivalent to Q' ($Q \sim Q'$).

LEMMA 6. *If $Q(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2)$ then $Q_{T|H_1}$ has the phase portrait of type f from Proposition* 2.

PROOF. To obtain the behavior of Q_T, we study the vector fields R_Q, S_Q and W_Q; considering that Q_T is symmetric these vector fields will give us the phase portrait of Q_T on the sets $\{(x_1, x_2, x_3) \in S^2 | x_i \neq 0\}$, $i = 1, 2, 3$, respectively. In this example,

$$R_Q(1, x_2, x_3) = (0, x_2 - x_2^2, x_3 - x_3^2),$$

$$S_Q(x_1, 1, x_3) = (x_1 - x_1^2, 0, x_3 - x_3^2),$$

$$W_Q(x_1, x_2, 1) = (x_1 - x_1^2, x_2 - x_2^2; 0);$$

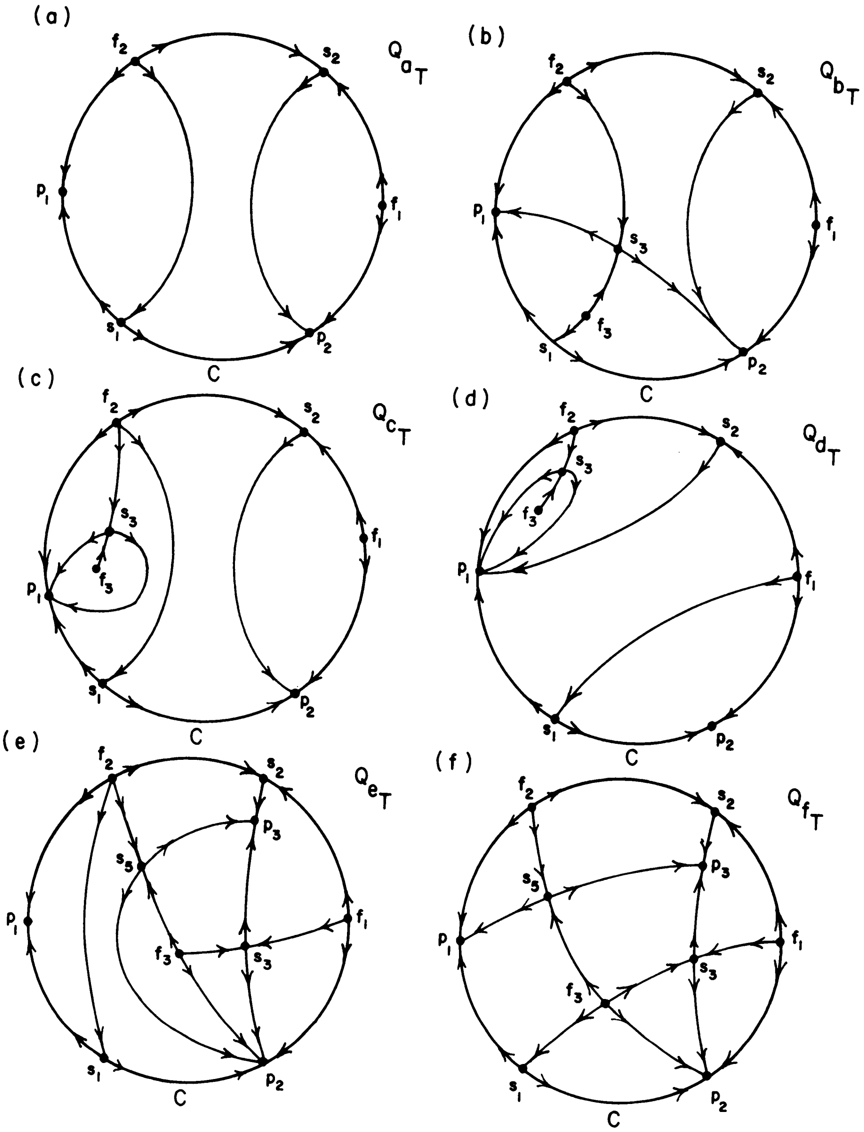

FIGURE 16

each one has 4 singularities and no closed orbits because they are gradient vector fields of the following functions, respectively:

$$f(x_2, x_3) = x_2^2/2 - x_2^3/3 + x_3^2/2 - x_3^3/3;$$
$$g(x_1, x_3) = x_1^2/2 - x_1^3/3 + x_3^2/2 - x_3^3/3,$$
$$h(x_1, x_2) = x_1^2/2 - x_1^3/3 + x_2^2/2 - x_2^3/3 \quad \text{(Figure 17).}$$

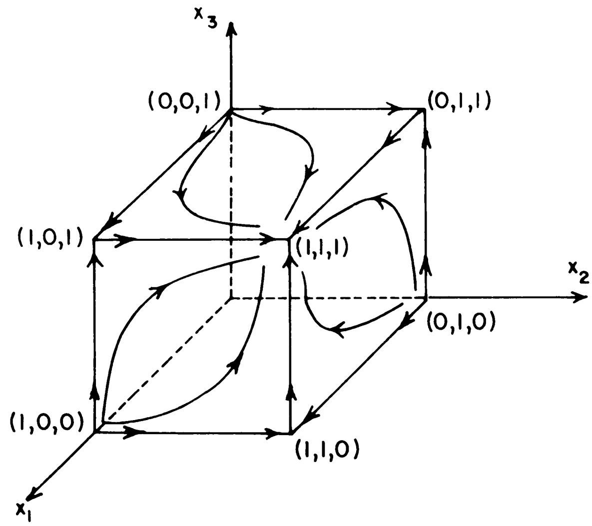

FIGURE 17

The singularity (1, 1, 1) is common to the vector fields R_Q, S_Q and W_Q. In Π_1 the vector field R_Q is Morse-Smale [7] with singularities (1, 0, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0); in Π_2 and Π_3 the behavior is similar. We can remark that, in this case for example, the curve C can be taken as the set $\{(x_1, x_2, 0) \in S^2\}$. The phase portrait of Q_T in H_1 is then the following: (Figure 18)

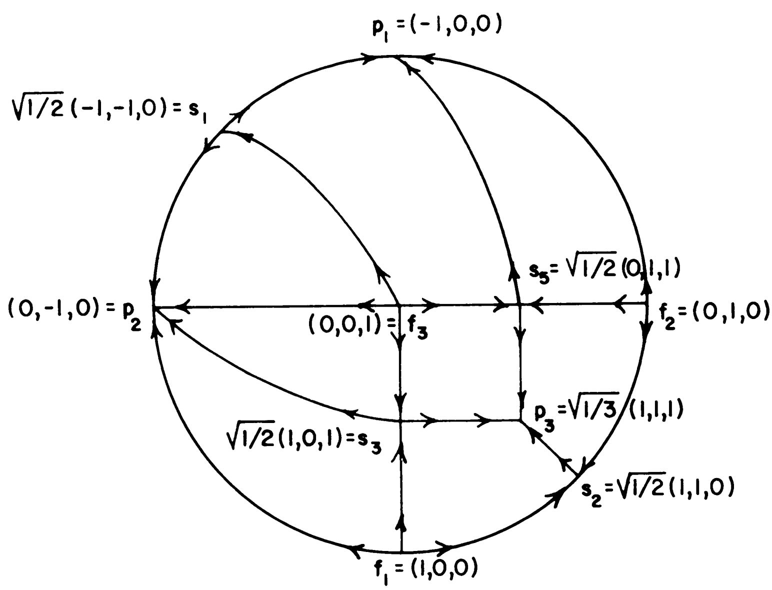

FIGURE 18

Using similar arguments as Lemma 6, the following propositions have immediate proof:

PROPOSITION 3. *If Q_T has two hyperbolic singularities and no closed orbits, then $Q \sim Q'$, where $Q'(x) = (x_1x_3, x_2x_3, -(x_1^2 + x_2^2 + \varepsilon x_3^2))$, $\varepsilon > 0$ small.*

PROPOSITION 4. *If the phase portrait of $Q_{T|H_1}$ is of type* (a) *in Proposition* 2, *i.e., if Q_T is a Morse-Smale vector field with* 6 *singularities and no closed orbits, then $Q \sim Q'$, where $Q'(x) = (x_2^2, x_1^2, x_3^2)$.*

PROPOSITION 5. *If Q_T is a Morse-Smale vector field with* 10 *singularities and no closed orbits, i.e., if the phase portrait of $Q_{T|\bar{H}_1}$ is type* (b), (c) *or* (d) *from Proposition* 2, *then $Q \sim Q_b'$, Q_c' or Q_d', respectively, where*:

$$Q_b'(x) = \left(x_1^2 + 2x_2x_3, x_2^2 + 2x_1x_3 + \varepsilon x_1x_2, x_3^2 + \varepsilon x_1x_3\right),$$

$$Q_c'(x) = \left(-x_1^2 - x_2^2 + 2\varepsilon x_1x_2 + x_2x_3, -\varepsilon x_1^2 - \varepsilon x_2^2 - 2x_1x_2 - x_1x_3, -\varepsilon x_3^2\right),$$

$$Q_d'(x) = \left(-x_1^2 - x_2^2 + 2\varepsilon x_1x_2 + x_2x_3,\right.$$
$$\left. -\varepsilon x_1^2 - \varepsilon x_2^2 - 2x_1x_2 - (1+\varepsilon)^2 x_1x_3 + \varepsilon x_2x_3, 0\right),$$

$\varepsilon > 0$ *small enough.*

PROPOSITION 6. *If Q_T is a Morse-Smale vector field with* 14 *singularities and no closed orbits, i.e., if the phase portrait of $Q_{T|\bar{H}_1}$ is type* (e) *or* (f) *from Proposition* 2, *then $Q \sim Q_e'$ or Q_f', respectively, where,*

$$Q_e'(x) = \left(-x_1^2 + 2x_1x_3 + x_2x_3, x_2^2, x_3^2\right),$$

$$Q_f'(x) = \left(x_1^2, x_2^2, x_3^2\right).$$

These propositions give us all the possible types of a vector field Q without closed invariant cones and such that Q_T is a Morse-Smale vector field; then:

THEOREM 2. *There are seven different topological classes of Morse-Smale vector fields Q_T with no limit cycles.*

We observe that for every Q such that Q_T is Morse-Smale and does not have closed orbit, $Q_{T|\bar{H}_1}$ is equivalent to some vector field contained in the cases (5.a) and (5.d) obtained in [**10**].

2.4. "*An example of a stable vector field Q that has two closed invariant cones*". Let $Q = (x_2x_3, -x_1x_3, x_1^2 + x_2^2 - x_3^2)$. We have

$$W_Q(x_1, x_2) = \left(x_1^2 + x_2^2 - 1\right)(x_1, x_2) + (x_2, x_1),$$

whose phase portrait is (Figure 19) and

$$R_Q(x_2, x_3) = \left(x_2^2x_3 + x_3, x_2x_3^2 - 1 - x_2^2 + x_3^2\right),$$

whose phase portrait is (Figure 20).

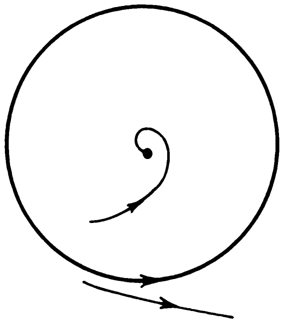

FIGURE 19

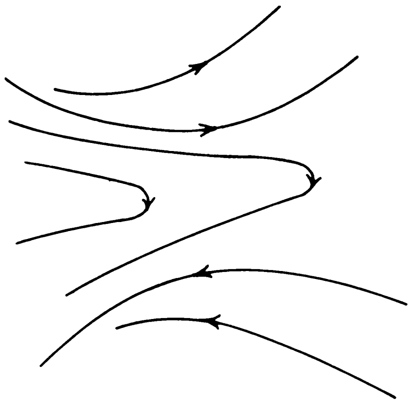

FIGURE 20

Using W_Q, R_Q and the central projection we find Q_T (Figure 21).

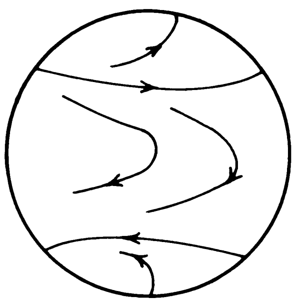

FIGURE 21

3. "A class of stable actions of the group of affine transformations of the real line". In this section we study the actions generated by the vector fields $-I$, Q in R^3 where I is the identity and Q is a polynomial homogeneous vector field of degree 2. The main results follow basically from the theorems of §1 on homogeneous vector fields.

3.1. *Local G^2-actions.* Here, we will study the topological structure of the local G^2-actions in R^n, generated by two vector fields X and Y such that $[X, Y] = Y$, where X has a singularity $0 \in R^n$ that is a hyperbolic attractor. We say that ϕ is a local action if there are neighborhoods $e \in U \subset G^2$, $0 \in V \subset R^n$, $0 \in W \subset R^n$, such that ϕ: $U \times V \to W$:

(1) $\phi(e, x) = x$;

(2) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ where $x \in V$, $\phi(g_2, x) \in V$, $g_1 g_2 \in U$.

These actions induce naturally a vector field $T_{(X,Y)}$ on a sphere in V, that by abuse of language we will call S^{n-1}, that is defined by

$$T_{(X,Y)}(x) = -\langle Y(x), I(x)\rangle X(x) + \langle X(x), I(x)\rangle Y(x), \qquad x \in S^{n-1},$$

where $\langle\ ,\ \rangle$ is the usual inner product in R^n. By definition $T_{(X,Y)}(x) \in T_x S^{n-1}$ for every $x \in S^{n-1}$; this vector field is tangent to the orbits of the action generated by

X and Y. We intend to prove here Proposition 1, i.e., an action under these conditions is structurally stable provided that the vector field $T_{(X,Y)}$ is structurally stable. Notice that

(1) the singularities of $T_{(X,Y)}$ are the intersections of the singular orbits of dimension 1 of the action with the sphere S^{n-1};

(2) if $0 \in R^n$ is an isolated singularity of X in R^n, then $0 \in R^n$ will be an isolated singularity of the action.

To see this, it is enough to verify that $Y(0) = 0$. In a neighborhood of zero, we have $[X, Y] = Y$ if and only if

$$X_s \circ Y_t = Y_{te^s} \circ X_s, \qquad (*)$$

for $t \in I_1 \subset R$, $s \in I_2 \subset R$, such that $I_1 \times I_2 \subset U$. By contradiction suppose that $Y(0) \neq 0$; then there exists a ball $B(0, \delta_0)$ of center zero and ray δ_0 in R^n whose boundary is transverse to X, and $t_0 \in I_1$, $t_0 \neq 0$, such that $Y_{t_0}(0) \in \partial B(0, \delta_0)$ and for $\varepsilon > 0$ small enough $t = (t_0 + \varepsilon) \in I_1$, $Y_t(0) \notin B(0, \delta_0)$. From $(*)$ we have that $X_s Y_t(0) = Y_{te^s}(0)$, i.e., the orbit of Y through the point 0 is invariant by the flow of X. It follows that there is $s_0 \in I_2$, $s_0 > 0$ such that $Y_{t_0e^{s_0}}(0) \notin B(0, \delta_0)$, whereas $X_{s_0} \circ Y_{t_0}(0) \in B(0, \delta_0)$; this is absurd by $(*)$. So $Y(0) = 0$.

We will call $\rho_{(X,Y)}$ the local G^2-action generated by the vector fields X and Y.

Proposition 1. *If $T_{(X,Y)}$ is a vector field C^r-structurally stable in S^{n-1}, then $\rho_{(X,Y)}$ is C^r-structurally stable.*

Proof. Let $\rho_{(\bar{X},\bar{Y})}$ be an action C^r close to $\rho_{(X,Y)}$. By definition the vector field $T_{(\bar{X},\bar{Y})}$ in the sphere is C^r close to $T_{(X,Y)}$; then there exists a homeomorphism $\bar{h}$ in S^{n-1} that is a topological equivalence between the flow of $T_{(X,Y)}$ and $T_{(\bar{X},\bar{Y})}$. We define the following homeomorphism h in R^n: $h(0) = \bar{0}$ where $\bar{0}$ is the hyperbolic sink of $\bar{X}$ and $h(x) = \bar{X}_{-t} \circ \bar{h} \circ X_t(x)$, $x \neq 0$; X_t and $\bar{X}_t$ are the flows of X and $\bar{X}$, respectively, and $X_t(x) \in S^{n-1}$ for some $t \in R^n$.

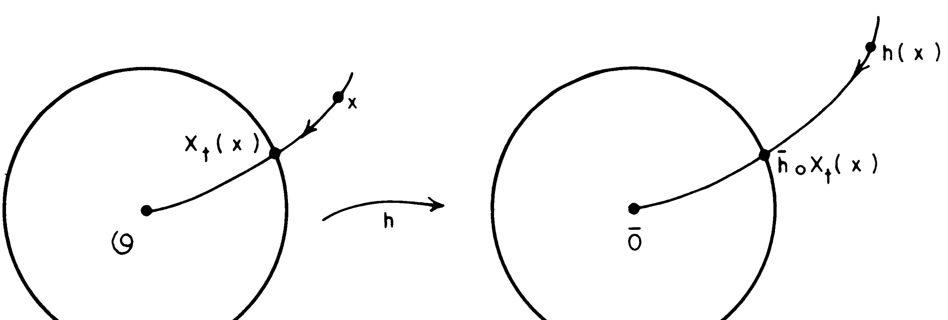

FIGURE 22

The homeomorphism h is a topological equivalence between the actions $\rho_{(X,Y)}$ and $\rho_{(\bar{X},\bar{Y})}$.

3.2. *Existence of actions C^2-structurally stable generated by vector fields $I/(1-k)$, Q, where Q is homogeneous of degree $k \geqslant 2$.* It is easy to see that $[I/(1-k), Q] = Q$, for every homogeneous Q of degree $k \geqslant 2$; then the vector fields $(I/(1-k), Q)$ generate local G^2-actions. As a corollary of Proposition 1 we have that, if the vector field $T_{(I/(1-k),Q)} = Q_T$ is structurally stable then $\rho_{(I/(1-k),Q)} = \rho_Q$ is structurally stable.

THEOREM A. *The local G^2-actions generated by the vector fields $(I/(1-2t), Q)$ where*

$$Q(x_1, x_2, \ldots, x_n) = \left(x_1^{2t}, x_2^{2t}, \ldots, x_n^{2t}\right), \qquad t \in N, \quad t \geq 1, \quad n > 3,$$

is C^r-structurally stable, $r \geq 2t$, and has a fixed point that is $0 \in R^n$, $2\sum_{p=1}^{n}\binom{n}{p}$ singular orbits of dimension 1, *and the orbits of dimension* 2 *are all homeomorphic to* R^2.

THEOREM B. *The local G^2-actions generated by the vector fields $(I/(-2t), Q)$ where $Q(x_1, x_2, \ldots, x_n) = (x_1^{2t+1}, x_2^{2t+1}, \ldots, x_n^{2t+1})$, $t \in N$, $t \geq 1$ and $n \geq 3$ is C^r-structurally stable, $r \geq 2t+1$, and has a fixed point that is $0 \in R^n$, $\sum_{p=1}^{n}\binom{n}{p}2^p$ singular orbits of dimension* 1 *and the orbits of dimension* 2 *are all homeomorphic to* R^2.

The proofs of Theorems A and B are by induction on the dimension n using the fact that Q_T is a gradient like the Morse-Smale vector field and Proposition 1. For example, if $Q(x) = (x_1^3, x_2^3, x_3^3)$ then the phase space of Q_T is the following:

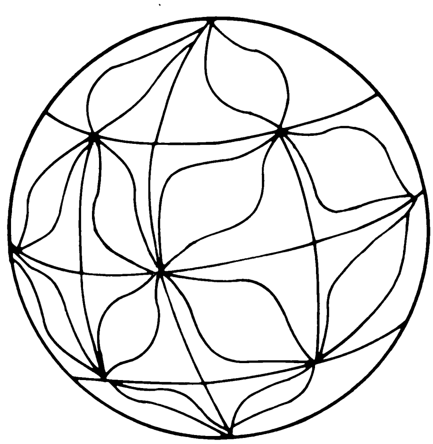

FIGURE 23

3.3. *Density of the stable G^2-actions generated by $(-I, Q)$, where Q is homogeneous of degree two.* Let $A = \{\rho_Q\colon G^2 \to \text{Diff } R^3$, generated by $(-I, Q)$, Q homogeneous of degree 2$\}$ and $\Sigma_A = \{$The actions belonging to A that are structurally stable$\}$.

THEOREM 3. *There is a subset $\Sigma_A \subset A$ dense in A whose actions are locally C^2-structurally stable.*

PROOF. Let ρ_Q be an arbitrary element of A and Q_T the vector field induced in S^2 by this action; using Theorem 1 we know how to approximate Q_T by a vector field $\overline{Q}_T$ that is Morse-Smale. If we take the action $\rho_{\overline{Q}}$ generated by $(-I, \overline{Q})$, this is close enough to ρ_Q and is structurally stable by Proposition 1 in this section.

3.4. *Classification of the G^2-actions structurally stable, generated by the vector fields $(-I, Q)$, without cylindrical orbits.* It is easy to see that the invariant cones of the polynomial homogeneous vector fields are the orbits of the action ρ_Q. Then the following theorem follows immediately from Theorem 2.

THEOREM 4. *There are seven different topological classes of G^2-actions belonging to Σ_A whose orbits are simply connected.*

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